Gen. Physiol. Biophys. (1985), **4**, 3—14

Single-File Diffusion of Uncharged Particles

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Abstract. An equation for unidirectional fluxes of nonelectrolytes and for the total flux in the single-file transport through a narrow pore has been derived. The equation obtained accounts for the correlations of the population of the pore in the coordinate of transport. The problem has been solved using superposition approximation and unidirectional fluxes has been found. The population profile in the pore was shown to have nonlinear shape; this is principally different from the results of the classical diffusion approach.

Key words: Channel — Single-file diffusion — Unidirectional flux — Population profile — Pair correlation function

Introduction

In many biological and physico-chemical systems, particles are transported through narrow pores or specialized channels; i. e. the transport process is of the single-file nature. The transport through selective channels in biological membranes is a good example of such a transport process. Obviously particle transport in such systems cannot be described by the classical diffusion equation since the latter is based on the idea of free diffusion and does not involve any limitations imposed by the single-file nature of the transport.

To describe the process of single-file transport, a special formalism is required. In the "discrete" approach, developed by Heckmann (1965, 1972), Aityan and Chizmadzhev (1973), Chizmadzhev and Aityan (1977), Aityan and Kalandadze (1977), Kohler and Heckmann (1979), Chizmadzhev and Aityan (1982), Markin and Chizmadzhev (1974), the particle transport has been represented as a series of successive jumps over a system of potencial barriers. The system has been described by the total state function characterizing probabilities of all the states of the system as a whole. However, for purely calculational reasons, the number of successive energy barriers to be overcome by a particle in the channel was considered small, being approximately 2 to 4. Obviously, in real systems the number of barriers is sufficiently large, and the discrete approach (considering a small number of barriers) can be valid only if there are a few comparatively high barriers in the system. If this is not the case the discrete approach to the multibarrier system faces quite complicate mathematical problems. These limita-

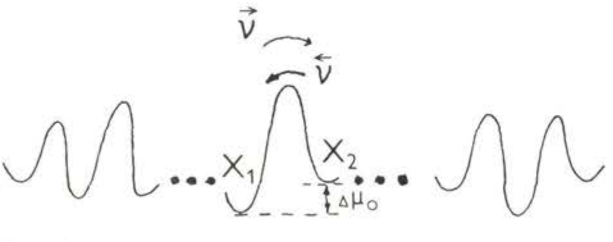

Fig. 1. Scheme of the model.

tions do not hold in some special cases only such as a practically empty channel (with a single particle in it) or a completely occupied one (Kohler and Heckmann 1979).

To solve the transport problem in a system with a large number of nearly identical barriers we have attempted deriving a differential equation for single-file diffusion which would account for correlations in the channel population and, first of all, for the single-file character of the transport itself. To simplify the problem we have excluded the effects of the electric field by considering the transport of uncharged particles.

Derivation of single-file diffusion equation

We suppose that in the transport of nonelectrolytes there are no long-range interactions between the particles and only a short-range action occurring as the competition of hard spheres for a vacant potential wells. Then to describe the transport through any potential barrier it is sufficient to know only the states of wells on both sides of the barrier. Let $F(Y_1 Y_2)$ be the pair correlation function (or pair state function) which represents the probability (in assembly) of the state $(Y_1 Y_2)$ of two adjacent wells in a given part of the channel. Y_1 and Y_2 may take values 1 or 0 depending upon whether the corresponding well is occupied or vacant (Fig. 1). Then the particle flux between these wells can be written as

$$j = \vec{v}F(10) - \overleftarrow{v}F(01) \tag{1}$$

where j is the flux from the left to right, and $\vec{v}$ and $\overleftarrow{v}$ are the rate constants of the jumps from the left to the right well and vice versa. Obviously, the rate constants are related by

$$\overleftarrow{v} = \vec{v} \exp(\Delta\mu_0) \tag{2}$$

where $\Delta\mu_0 = \mu_2^0 - \mu_1^0$ is the dimensionless difference between standard chemical potentials in the second and first of that adjacent potential wells respectively (Fig. 1). Noting that

$$\begin{aligned} F(10) + F(11) &= \Theta_1 \\ F(01) + F(11) &= \Theta_2 \end{aligned} \tag{3}$$

where Θ_1 and Θ_2 are the populations (probabilities of the occupation of the respective potential well) of the first and second potential well respectively, and assuming that the macroscopic parameters within the channel change smoothly, we obtain from Eq. (1) (Chizmadzhev and Aityan 1982).

$$j = -D\frac{\mathrm{d}\Theta}{\mathrm{d}x} - \frac{D}{2}[F(10) + F(01)]\frac{\mathrm{d}\mu_0}{\mathrm{d}x} \tag{4}$$

where $D = v\delta$ and δ is the distance between the adjacent wells, x is the transport coordinate.

It appears from Eq. (4) that the inclusion of correlations has affected only the term $\frac{D}{2}[F(10) + F(01)]\frac{\mathrm{d}\mu_0}{\mathrm{d}x}$. In spite of the inclusion of correlations of populations within the channel, the diffusion term $\mathrm{D}\frac{\mathrm{d}\Theta}{\mathrm{d}x}$ remained unchanged. Thus in a homogeneous channel with

$$\mu_0 = \text{const} \tag{5}$$

the flux is represented by the classical diffusion term

$$j = -D\frac{\mathrm{d}\Theta}{\mathrm{d}x} \tag{6}$$

Let us study more in detail the flux through the homogeneous channel ($\mu_0 = \text{const}$). Suppose the channel to link two solutions containing particles A and B with concentrations C_A and C_B respectively (Fig. 2). The left solution contains no particles B, while there are no particles A in the right one. The flux of particles A from the left to the right which is unidirectional flux from the left to the right can be expressed as

$$\begin{aligned} j_\mathrm{A} &= v_\mathrm{A}F(\mathrm{A}0) - v_\mathrm{A}F(0\mathrm{A}) = \\ &= v_\mathrm{A}[F(\mathrm{A}0) + F(\mathrm{AA}) + F(\mathrm{AB})] - v_\mathrm{A}[F(\mathrm{AA}) + F(\mathrm{AB})] - \\ &\quad - v_\mathrm{A}[F(0\mathrm{A}) + F(\mathrm{AA}) + F(\mathrm{BA})] + v_\mathrm{A}[F(\mathrm{AA}) + F(\mathrm{BA})] \end{aligned} \tag{7}$$

where in the pair correlation function $F(Y_1Y_2)$ the variables Y_i can take values A, B and 0 which show whether the respective well is occupied by particles of type

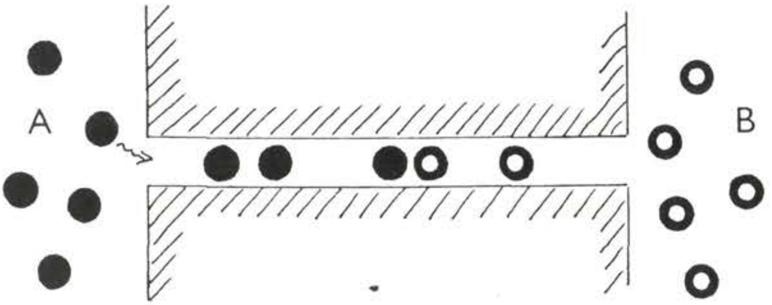

Fig. 2. Single-file transport of particles A and B.

A or B, or vacant. The rate constants are $\vec{\nu}_A = \overleftarrow{\nu}_A = \nu_A$ by virtue of the condition (5) since a homogeneous channel is considered. Note that $F(\mathrm{BA}) = 0$ since by virtue of the single-file nature of the transport and the absence of particles A in the right solution and particles B in the left one. Hence particle A cannot be within the channel on right of a particle B. Thus expression (7) yields

$$j_A = -D_A \frac{d\Theta_A}{dx} - D_A F(\mathrm{AB})/\delta \tag{8}$$

where $D_A = \nu_A \delta$ and Θ_A is the population of the well (with coordinate x) with particle A

$$\Theta_A = F(\mathrm{AO}) + F(\mathrm{AA}) + F(\mathrm{AB}) \tag{9}$$

Similarly the expression for the flux of particles B which is unidirectional flux from the left to the right can be written as

$$j_B = D_B \frac{d\Theta_B}{dx} - D_B F(\mathrm{AB})/\delta \tag{8'}$$

where j_B is directed from the right to the left. All of Θ_A, Θ_B and $F(Y_1 Y_2)$ are functions of transport coordinate x.

The net flux

$$j = j_A - j_B \tag{10}$$

is

$$j = -D_A \frac{d\Theta_A}{dx} - D_B \frac{d\Theta_B}{dx} - (D_A - D_B) F(\mathrm{AB})/\delta \tag{11}$$

Equation (11) is identical with the classical diffusion equation only if $D_A = D_B$. If particles A and B are different ($D_A \neq D_B$) the diffusion equation has a correction due to cross states such as (AB). It is interesting to stress that pair states such as (AA) and (BB) do not contribute to the correction of the classical diffusion equation. Generalizing Eq. (11) to the case when there is a set of particles A_i in the left solution and B_k in the right one we get

$$j = -\sum_{\alpha = A_i, B_k} D_\alpha \frac{d\Theta_\alpha}{dx} - \frac{1}{\delta} \sum_{A_i, B_k} (D_{A_i} - D_{B_k})\, F(A_i B_k) \tag{12}$$

Thus the single-file diffusion equation in the homogeneous channel ($\mu_0 = \text{const}$) is of the type (12) and reduces to the classical diffusion equation only if all the particles transportable through the channel have the same kinetic characteristics (all $D_\alpha = D$). Generally the diffusion equation (12) involves cross state functions $F(A_i B_k)$ (pair correlation functions of cross states $(A_i B_k)$) which can be in principle obtained from balance equations for the pair state functions like those used in the papers by Heckmann (1965, 1972), Aityan and Chizmadzhev (1973), Chizmadzhev and Aityan (1977), Aityan and Kalandadze (1977) and Chizmadzhev and Aityan (1977). Generally speaking these balance equations contain state functions of higher orders up to the total state function as in any kinetic theory.

For illustration, we consider unidirectional fluxes with A and B being different tracers of the same substance, i. e. $D_A = D_B = D$. Unidirectional fluxes, j_A and j_B, are represented by Eqs. (8) and (8′), and the net flux $j = j_A - j_B$ by the classical Eq. (6). Let there be equilibrium on the boundaries of the channel with the corresponding solution, i. e.

$$\begin{aligned} \Theta_A(0) &= \frac{\gamma C_A}{\gamma C_A + 1}; \qquad & \Theta_A(l) &= 0; \\ \Theta_B(0) &= 0; \qquad & \Theta_B(l) &= \frac{\gamma C_B}{\gamma C_B + 1} \end{aligned} \tag{13}$$

and for the total population

$$\Theta = \Theta_A + \Theta_B \tag{14}$$

we have

$$\Theta(0) = \frac{\gamma C_A}{\gamma C_A + 1}; \qquad \Theta(l) = \frac{\gamma C_B}{\gamma C_B + 1} \tag{15}$$

where l is the channel length and γ is the partition coefficient between the channel and the solution. Then, according to Eq. (6) and the boundary conditions (15) the steady-state profile of the total population within the channel has the linear form

$$\Theta(x) = \Theta(0) + \frac{\Theta(l) - \Theta(0)}{l}\, x \tag{16}$$

Let us represent the pair correlation function F(AB) in Eqs (8) and (8′) as the product of the population (superposition approximation)

$$F(\mathrm{AB}) = \Theta_{\mathrm{A}}\Theta_{\mathrm{B}} \tag{17}$$

Then Eq. (8) would take the form

$$J_{\mathrm{A}} = -\frac{\mathrm{d}\Theta_{\mathrm{A}}}{\mathrm{d}X} - \Theta_{\mathrm{A}}\Theta_{\mathrm{B}} \tag{18}$$

where

$$J_{\mathrm{A}} = j\delta/D\,; \qquad X = x/\delta \tag{19}$$

or taking into account the relation (14),

$$J_{\mathrm{A}} = -\frac{\mathrm{d}\Theta_{\mathrm{A}}}{\mathrm{d}X} - \Theta_{\mathrm{A}}(\Theta - \Theta_{\mathrm{A}}) \tag{20}$$

where Θ is the already known function (16).

Results

(I) *Symmetrical case*

Eq. (20) may easily be integrated with the inclusion of the boundary conditions (13) and the known function (16) in the symmetrical case, i. e. at $\Theta_{\mathrm{A}}(0) = \Theta_{\mathrm{B}}(L)$. The expression for the unidirectional flux is obtained in the implicit form

$$\frac{R + \Theta}{R - \Theta} = \exp\,(LR) \tag{21}$$

where

$$\Theta = \frac{\gamma C_{\mathrm{A}}}{\gamma C_{\mathrm{A}} + 1}; \qquad R = \sqrt{\Theta^2 - 4J_{\mathrm{A}}}\,; \qquad L = l/\delta \tag{22}$$

From the implicit expression (21) the flux J_{A} can be found graphically as shown in Fig. 3, where the respective dependences of the right and left sides of Eq. (21) on R are plotted. The unidirectional flux can be obtained by the ordinate of the intersection point of these curves, R^*

$$J_{\mathrm{A}} = \frac{(R^* - \Theta)^2 - \Theta^2}{4} = \frac{R^*}{4}(R^* - 2\Theta) \tag{23}$$

From the analysis of Eq. (21) it may be seen that, at low concentrations, $C_{\mathrm{A}} \to 0$ (i. e. $\Theta \to 0$)

$$J_{\mathrm{A}} \sim \Theta/L \tag{24}$$

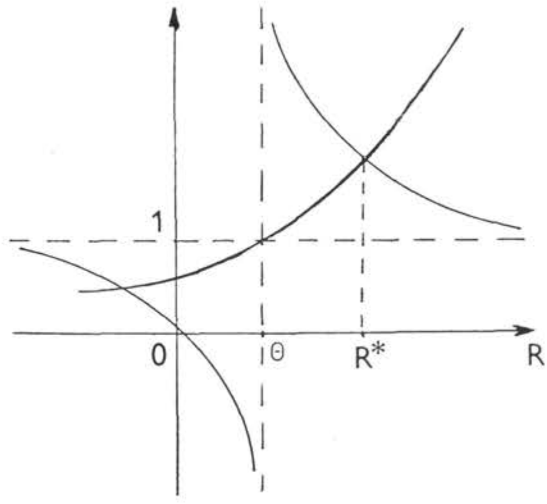

Fig. 3. Graphical solution to Eq. (20). Abscissa variable: R; ordinate: arbitrary units. The solution is found from the intersection of the plots for the left and right sides of Eq. (20).

i. e. the flux tends to the classical limit; in the limit of high concentrations, $C_A \to \infty(\Theta \to 1)$

$$J_A = \frac{1}{e^L - 1} - \frac{e^L(L-2)+2}{(e^L-1)^2}(1-\Theta) \tag{25}$$

Thus at high concentrations the unidirectional flux tends to its limiting value

$$J_A = \frac{1}{e^L - 1} \tag{26}$$

which is a function of the characteristic channel length, L.

It is also interesting to follow the dependence of the unidirectional flux on the channel length. If $L \to 0$

$$J_A = \frac{\Theta}{2} + \sqrt{\frac{2\Theta}{2}} \to \infty \tag{27}$$

and if $L \to \infty$

$$J_A \sim e^{-LR} \tag{28}$$

The distinction between the single-file diffusion and the classical free diffusion is also pronounced in the channel population profile. In the case of free diffusion,

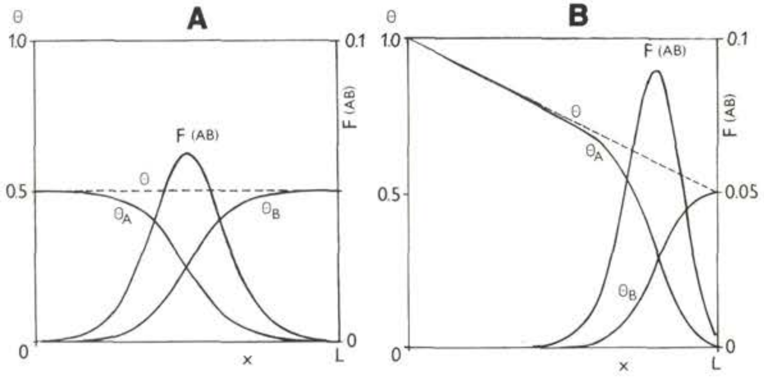

Fig. 4. Profiles of channel populations Θ_A, Θ_B, $\Theta = \Theta_A + \Theta_B$ and of the pair correlation function F(AB) in the coordinate of transport at: A — $\Theta_A(0) = \Theta_B(L) = 0.5$; B — $\Theta_A(0) = 1$, $\Theta_B(L) = 0.5$.

the profile of the channel population for each tracer, Θ_A and Θ_B, as well as the total profile, Θ, were linear in the coordinate whereas in the present case of the single-file transport

$$\Theta_A = \frac{\Theta}{2} + \frac{\sqrt{\Theta^2 + 4J_A}}{2} \cdot \frac{1 - \varkappa}{1 + \varkappa} \tag{29}$$

where

$$\varkappa = \frac{R - \Theta}{R + \Theta} \exp(XR) \tag{30}$$

and the flux, J_A, satisfies Eq. (25). The same applies to the profile Θ_B. Fig. 4 shows the profiles Θ_A, Θ_B, $\Theta = \Theta_A + \Theta_B$ and the pair correlation function $F(\text{AB})$. As evident from the above analysis, even under symmetrical conditions and superposition approximation, $F(\text{AB}) = \Theta_A \Theta_B$, the single-file transport would considerably differ from the classical diffusion transport and maximum blocking of the channel due to pair correlation function, F(AB) occurs in the central part of the channel (Fig. 4*A*). However, only at low populations, Θ_A, Θ_B, $\Theta \to 0$, the population profiles tend to the linear ones which corresponds to the classical diffusion representation.

(II) *Asymmetrical case*

In the asymmetrical case, i. e. at $\Theta_A(0) \neq \Theta_B(L)$ Eq. (20) cannot be analytically integrated. Therefore, we shall analyze the behaviour of the system using the

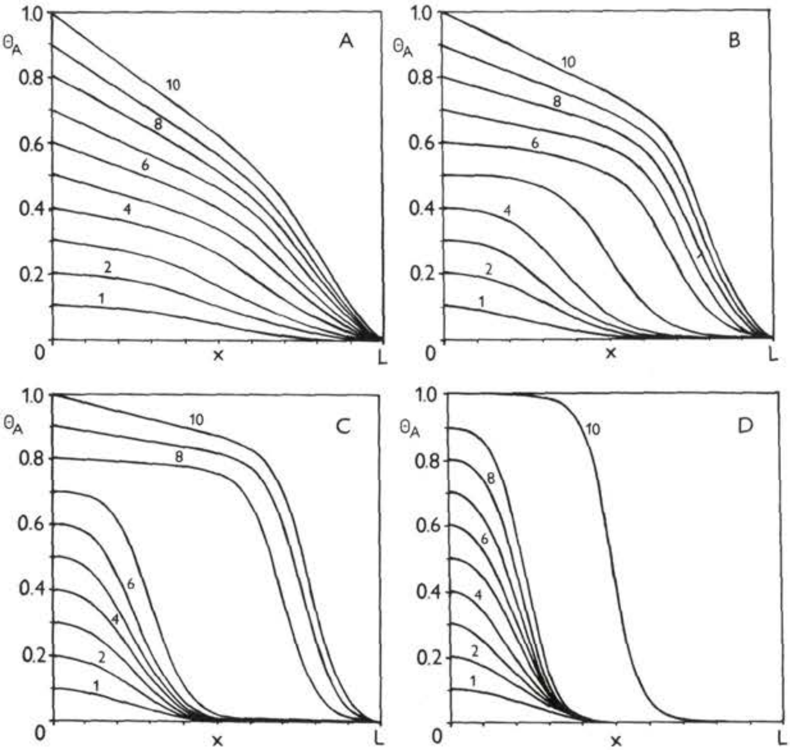

Fig. 5. Profiles of the channel populations Θ_A for: A — $\Theta_B(L)=0.25$; B — $\Theta_B(L)=0.5$; C — $\Theta_B(L)=0.75$; D — $\Theta_B(L)=1$ at various $\Theta_A(0)=0.1$ (curves 1); 0.2 (curves 2); 0.3 (curves 3); 0.4 (curves 4); 0.5 (curves 5); 0.6 (curves 6); 0.7 (curves 7); 0.8 (curves 8); 0.9 (curves 9); 1.0 (curves 10).

numerical solution to Eq. (20) with boundary conditions (15). As an example Fig. 4B shows the profiles of populations, Θ_A, Θ_B, and the pair correlation function, $F(\mathrm{AB})$, at the boundary conditions

$$\Theta_A(0)=1; \qquad \Theta_A(L)=0;$$
$$\Theta_B(0)=0; \qquad \Theta_B(L)=0.5.$$

Fig. 5 ($A-D$) shows the population profiles, Θ_A, for $\Theta_B(L)=0.25$; 0.5; 0.75 and 1, respectively, at various $\Theta_A(0)$. It appears from these figures that the channel population profiles have nonlinear shapes which makes them qualitatively different from the channel population profiles obtained from the classical diffusion representation. Fig. 5A also shows that profiles Θ_A slightly differ from the linear ones at low $\Theta_B(L)$. With increasing $\Theta_B(L)$ the unidirectional fluxes J_A and J_B start to block each other which results in a strong nonlinearity of profiles Θ_A and Θ_B (Fig.

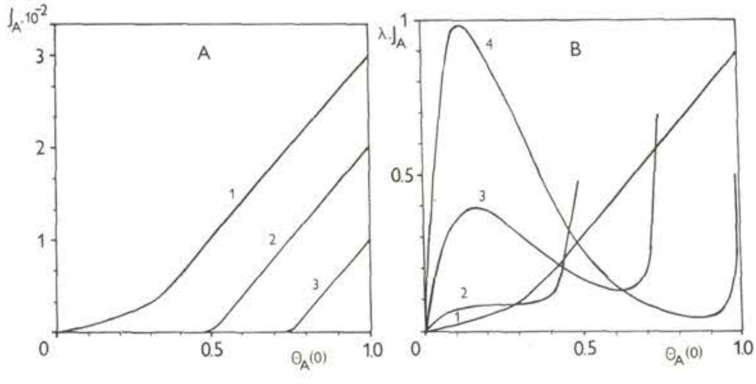

Fig. 6. Dependence of the unidirectional flux J_A on the boundary population $\Theta_A(0)$ at various $\Theta_B(L) = 0.25$ (curves 1); 0.5 (curves 2); 0.75 (curves 3); 1 (curves 4): *A* — all the curves are shown in the same "large" scale to illustrate the threshold-like behaviour of the unidirectional fluxes; *B* — curves are shown in various "small" scales to illustrate the nonlinear behaviour of the unidirectional fluxes. Scaling factor $\lambda = 30$ (curve 1), 5×10^6 (curve 2), 1×10^5 (curve 3), 1×10^3 (curve 4).

$5B - D$). The dependence of unidirectional fluxes, J_A, J_B, on the boundary concentrations C_A, C_B (or $\Theta_A(0)$, $\Theta_B(L)$) is of particular interest. Fig. 6*A* shows the dependences of J_A on $\Theta_A(0)$ at various $\Theta_B(L)$. It is clear that the unidirectional flux J_A decreases almost linearly with increasing $\Theta_A(0)$, and practically disappears at $\Theta_A(0) \approx \Theta_B(L)$. This threshold-like behaviour of the unidirectional flux J_A is less pronounced at small $\Theta_B(L)$ only. Such a behaviour of the unidirectional flux can readely be explained. Under equal boundary conditions, $\Theta_A(0) = \Theta_B(L)$, the net flux $J = J_A - J_B$ is zero while the unidirectional flux $J_A = J_B$ is determined by the degree of the blocking of the channel, i. e. by the pair correlation function $F(AB)$. Here, the higher $\Theta_A(0)$ and $\Theta_B(L)$ the smaller the unidirectional fluxes. At $\Theta_A(0) > \Theta_B(L)$ the net flux is

$$J = \frac{\Theta_A(0) - \Theta_B(L)}{L}$$

being mainly determined by the transport of particles of the type A, i. e. $J \approx J_A$. At $\Theta_A(0) = \Theta_B(L)$ the flux J_A disappears similary to the unidirectional flux under symmetrical conditions. Thus at a high population of the boundary well, $\Theta_B(L)$, when the symmetrical unidirectional fluxes are small owing to strong blocking effects, the dependence of J_A on $\Theta_A(0)$ is practically of the threshold-like behaviour (Fig. 6*A*, curve 3). At low populations of the boundary well, $\Theta_B(L)$, the symmetrical unidirectional fluxes are comparable to the linear ones, and no pronounced threshold-like dependence of J_A on $\Theta_A(0)$ is observed (Fig. 6*A*, curve 1).

The dependence of J_A on $\Theta_A(0)$ at $\Theta_A(0) < \Theta_B(L)$ is also of interest (Fig. 6*B*). It is evident that this dependence is nonmonotonic at high boundary populations $\Theta_B(L)$ and it becomes monotonic with decreasing $\Theta_B(L)$. This effect is accounted for by the fact that the unidirectional flux in this region is determined by both the boundary population level which increases the unidirectional flux, and the channel blocking by $F(AB)$ which decreases it. Operating in opposite ways, these two trends result in a nonlinear dependence of J_A on $\Theta_A(0)$. At low boundary populations $\Theta_B(L)$, the blocking in the region $\Theta_A(0) < \Theta_B(L)$ is not clearly defined (since F(AB is of the second order on Θ), which leads to a monotonic increase in the dependence curve of J_A on $\Theta_A(0)$.

Thus the unidirectional flux J_A is affected by three trends. At $\Theta_A(0) > \Theta_B(L)$, the unidirectional flux J_A, to within the symmetrical unidirectional flux, is equal to the total flux and is practically determined by the difference between these boundary populations. Here, at high boundary populations $\Theta_B(L)$ unidirectional fluxes are very small, resulting in a threshold-like dependence of the unidirectional flux J_A on the boundary population $\Theta_A(0)$ (see Fig. 6*A*). At $\Theta_A(0) < \Theta_B(L)$ the unidirectional flux is affected by two trends: one increasing, due to the concentration gradient, and another decreasing, due to blocking. These two trends lead to a nonmonotonic dependence of J_A on $\Theta_A(0)$. At low boundary populations $\Theta_B(L)$ the blocking, which is of the second order in Θ, is drastically diminished, and the dependence of J_A on $\Theta_A(0)$ becomes monotonic.

In the present paper we have derived an equation for single-file diffusion which accounts for the single-file nature of the transport ($F(BA) = 0$) and of the correlation of the channel population. To close the chain of equations we used the superposition approximation $F(AB) = \Theta_A \Theta_B$, which made it possible to solve the problem. Also in this first approximation, the unidirectional fluxes are significantly different from those obtained in the classical diffusion representation. However, it would be desirable to take accurate account of the pair and higher correlations within the channel and to investigate the effect of the electric field on the transport of charged particles. These topics will be discussed in our future communications.

The result of this theoretical approach can be used for interpretation of tracer transport experiments in biological and artificial membranes or in superionic conductors possess the single-file transport and can make the frame of a general interpretation of these phenomena.

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Received January 17, 1984 / Accepted May 14, 1984